JOURNAL OF APPROXIMATION THEORY 33, 85–95 (1981)

A Note on Mean Convergence of Lagrange Interpolation

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Communicated by G. Meinardus

Received October 24, 1979

1. INTRODUCTION

Let

$$-1 \leqslant x_{1,n} < x_{2,n} < \cdots < x_{n,n} \leqslant 1, \qquad n = 1, 2, \ldots, \tag{1}$$

be a matrix X of nodes and, for a function f defined in $[-1, 1]$ and each n, let $L_n(f, x)$ be the corresponding Lagrange interpolation polynomial of degree $\leqslant n-1$ based on the nodes (1).

Erdös and Turán have proved, in their paper [3], that for every continuous function f on $[-1, 1]$ and every sequence of polynomials $\{\omega_n\}$ (deg $\omega_n = n$) satisfying

$$\int_{-1}^{1} w(x) \cdot \omega_n(x) \cdot \omega_m(x)\, dx = \delta_{n,m},$$

$$\int_{-1}^{1} w(x)\, dx < +\infty, \qquad w(x) \geqslant 0 \ \forall\, x \in (-1, 1),$$

the condition

$$\lim_n \int_{-1}^{1} w(x)\{L_n(f, x) - f(x)\}^2\, dx = 0 \tag{2}$$

holds for all matrices formed by the roots of the polynomials

$$R_n(x) = \omega_n(x) + A_n \omega_{n-1}(x) + B_n \omega_{n-2}(x), \qquad B_n \leqslant 0, \tag{3}$$

provided each $R_n(x)$ has n distinct roots in $[-1, 1]$.

In particular, (2) is fulfilled for the matrices formed by the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ ($w(x) = (1-x)^\alpha(1+x)^\beta$, α and $\beta > -1$) and of the polynomials $(1-x^2)(d/dx)\, P_n^{(\alpha,\beta)}(x)$ which are of type (3).

0021-9045/81/100085-11$02.00/0

It is well known that condition (2) for matrix (1) guarantees the convergence of certain collocation-projection procedures for the approximate solutions of some operator equations (see Vainikko [6]).

In some extended collocation procedures (see [1]), it is useful to answer the following question:

Find a matrix Y of additional nodes

$$-1 \leqslant y_{1,n} < y_{2,n} < \cdots < y_{m(n),n} \leqslant 1, \qquad x_{i,n} \neq y_{j,n} \ \forall\, i, j, n, \tag{4}$$

where $m(n) = an + b$, a, b *real and* $a > 0$, *such that, for the new matrix* $X \cup Y$:

$$x_{1,n}, \ldots, x_{n,n}, y_{1,n}, \ldots, y_{m(n),n}, \qquad n = 1, 2, \ldots, \tag{5}$$

the Lagrange interpolation polynomials $L_{n+m(n)}(f, x)$ *based on the nodes* (5) *satisfy*

$$\lim_{n} \int_{-1}^{1} w(x) \{L_{n+m(n)}(f, x) - f(x)\}^2 \, dx = 0 \tag{6}$$

for some class of functions f defined on $[-1, 1]$.

If the matrix X is formed by the roots of $U_n(x)$, then since $U_{2n+1} = nU_n(x)\, T_{n+1}(x)$, the additional matrix Y may be formed by the roots of $T_{n+1}(x)$ and, quite obviously, (6) holds for any continuous function f.

Unfortunately, this is the only known case of an orthogonal polynomial P which can be factorized into two polynomials, one of which is in the same orthogonality class as P. However, with some restriction on f, some positive result can be found for a large class of nodes matrices.

We shall prove the following theorem:

THEOREM A. *Let the* $x_{i,n}$*'s nodes of the matrix* (5) *satisfy the condition*

$$(\mathrm{E}) \quad \begin{cases} \text{if } n(\beta_n - \alpha_n) \to \infty,\ 0 \leqslant \alpha_n < \beta_n \leqslant \pi \text{ then} \\ \overline{\lim_{n\to\infty}} \dfrac{N_n(\alpha_n, \beta_n)}{n(\beta_n - \alpha_n)} \leqslant \dfrac{1}{\pi} \qquad \text{and} \qquad \underline{\lim_{n\to\infty}} (\vartheta_{i+1,n} - \vartheta_{i,n}) n > 0 \qquad \forall\, i, \end{cases}$$

where $x_{i,n} = \cos \vartheta_{i,n}$ *and* $N_n(\alpha, \beta)$ *stands for the number of* $\vartheta_{i,n}$*'s in* $[\alpha, \beta]$.

Let the additional nodes $y_{i,n}$ $(i = 1, 2, \ldots, an + b)$ *form a strongly normal matrix. Then, using the notation*

$$\Phi_n(x) = \prod_{i=1}^{n} (x - x_{i,n}), \tag{7}$$

$$M(n) = \frac{\int_{-1}^{1} w(x)\, \Phi_n^2(x)\, dx}{\min_i \Phi_n^2(y_{i,n})}, \tag{8}$$

condition (6) *is fulfilled for every function f such that*

$$\lim_{n\to\infty} n \cdot E_n^2(f) \cdot M(n) = 0, \tag{9}$$

where $E_n(f)$ *is the best uniform approximation of* f *in* $[-1, 1]$ *by polynomials of degree* $\leqslant n$.

As an application of Theorem A, we obtain the following corollaries concerning the classical Jacobi polynomials $P_n^{(\alpha)}(x) = P_n^{(\alpha,\alpha)}(x)$.

COROLLARIES 1, 2, 3. *If matrix* (5) *is formed by the roots of the polynomials*

$$\text{(j)}\quad Q_{n'}(x) = (1-x^2)\, P_n^{(\alpha)}(x) \frac{d}{dx} P_n^{(\alpha)}(x), \qquad -1 < \alpha < 1,$$

or

$$\text{(jj)}\quad Q_{n'}(x) = P_n^{(\alpha)}(x) \frac{d}{dx} P_n^{(\alpha)}(x), \qquad -1 < \alpha < 0,$$

then the Lagrange interpolation polynomials $L_n(f, x)$ *based on the roots of* $Q_n(x)$ *satisfy*:

in case (j),

$$\lim_{n\to\infty} \int_{-1}^{1} (1-x^2)^{\alpha} \{L_{n'}(f, x) - f(x)\}^2 \, dx = 0$$

$$\textit{if}\quad \left\{0 \leqslant \alpha < 1, f \in \mathrm{Lip}\,\gamma, \gamma > \frac{1}{2};\ \textit{or}\ -1 < \alpha < 0, f \in \mathrm{Lip}\,\gamma, \gamma > -\alpha\right\};$$

in case (jj),

$$\lim_{n\to\infty} \int_{-1}^{1} (1-x^2)^{\alpha+1} \{L_{n'}(f, x) - f(x)\}^2 \, dx = 0,$$

$$-1 < \alpha < 0, f \in \mathrm{Lip}\,\gamma, \gamma > \frac{1}{2}.$$

2. PRELIMINARIES AND PROOF OF THEOREM A

In his paper [2], Erdös proved that, in order for the matrix X to satisfy condition (E), it is necessary and sufficient that for every positive number c

there exists a sequence of operators $p_n(f, c, x)$, defined for all $f \in C[-1, 1]$, such that

(i) $p_n(f, c, x)$ is a polynomial of degree $\leqslant n(1+c)$ $\quad (n = 1, 2, \ldots)$.

(ii) $p_n(f, c, x_{i,n}) = f(x_{i,n}) \quad (i = 1, \ldots, n)$,

(iii) $\lim_{n\to\infty} \| p_n(f, c, x) - f(x) \| = 0$ $\quad$ for every $f \in C[-1, 1]$.

As far as the rate of convergence in (iii) is concerned, Freud has proved in [4] that condition (E) is also necessary and sufficient for the existence, for every positive constant c, of a sequence of operators $p_n(f, c, x)$ satisfying (i), (ii) and

(iv) $\| p_n(f, c, x) - f(x) \| \leqslant K(c)\, E_n(f)$ $\quad$ for every $f \in C[-1, 1]$.

By using notations (7), (8) and

$$\Psi_n(x) = \prod_{i=1}^{m(n)} (x - y_{i,n}),$$

$$1_i(x) = \frac{\Psi_n(x)}{\Psi_n'(y_{i,n})(x - y_{i,n})}$$

let us prove the following lemma:

LEMMA 1. *Let X and Y be respectively the matrices* (1) *and* (4). *If the nodes of the matrix X are arbitrarily choosen and the matrix Y is strongly normal, then*

$$\sum_{i=1}^{m(n)} \int_{-1}^{1} w(x)\, \Phi_n^2(x) \frac{1_i^2(x)}{\Phi_n^2(y_{i,n})}\, dx \leqslant \frac{M(n)}{\rho} \qquad \rho \text{ positive const.}$$

Proof. Since Y is strongly normal, there exists a positive constant ρ such that

$$v_{i,n}(x) = 1 - \frac{\Psi_n''(y_{i,n})}{\Psi_n'(y_{i,n})}(x - y_{i,n}) \geqslant \rho > 0 \qquad i = 1, \ldots, m(n),$$

where, for the Hermite interpolation formula,

$$\sum_{i=1}^{m(n)} v_{i,n}(x)\, 1_i^2(x) = 1.$$

Therefore

$$\sum_{i=1}^{m(n)} 1_i^2(x) \leqslant \frac{1}{\rho},$$

$$\sum_{i=1}^{m(n)} w(x) \frac{\Phi_n^2(x)\, 1_i^2(x)}{\Phi_n^2(y_{i,n})} \Phi_n^2(y_{i,n}) \leqslant \frac{w(x)}{\rho} \Phi_n^2(x),$$

$$\sum_{i=1}^{m(n)} \int_{-1}^{1} w(x) \frac{\Phi_n^2(x)\, 1_i^2(x)}{\Phi_n^2(y_{i,n})}\, dx \leqslant \frac{1}{\rho} \frac{\int_{-1}^{1} w(x)\, \Phi_n^2(x)\, dx}{\min_i \Phi_n^2(y_{i,n})} = \frac{M(n)}{\rho}.$$

Proof of Theorem A. For a fixed positive c such that $c < a$, let $p_n(x) = p_n(f, c, x)$ be the polynomial satisfying the properties (i), (ii) and (iv). Since $(r+s)^2 \leqslant 2r^2 + 2s^2$ and $L_{n+m(n)}(f, x) - f(x) = L_{n+m(n)}(f, x) - p_n(x) + p_n(x) - f(x)$,

$$\int_{-1}^{1} w(x)\{L_{n+m(n)}(f, x) - f(x)\}^2\, dx$$
$$\leqslant 2 \int_{-1}^{1} w(x)\{L_{n+m(n)}(f, x) - p_n(x)\}^2\, dx$$
$$+ 2 \int_{-1}^{1} w(x)\{p_n(x) - f(x)\}^2\, dx.$$

From (iv),

$$\int_{-1}^{1} w(x)\{p_n(x) - f(x)\}^2\, dx$$
$$\leqslant K^2(c)\, E_n^2(f) \int_{-1}^{1} w(x)\, dx \to 0 \qquad \text{as} \quad n \to \infty.$$

If $L_{n+m(n)}(f, x)$ interpolates f over $n(1+a)+b$ nodes and therefore $L_{n+m(n)}(q, x) = q$ for every polynomial q of degree $\leqslant n(1+a)+b-1$, then, for n sufficiently large, $L_{n+m(n)}(p_n, x) = p_n(x)$.

Thus

$$\int_{-1}^{1} w(x)\{L_{n+m(n)}(f, x) - p_n(x)\}^2\, dx$$
$$= \int_{-1}^{1} w(x)\{L_{n+m(n)}(f - p_n, x)\}^2\, dx \tag{10}$$

where, because of (ii),

$$L_{n+m(n)}(f - p_n, x) = \sum_{i=1}^{m(n)} \frac{\Phi_n(x)\, 1_i(x)}{\Phi_n(y_{i,n})} (f - p_n)(y_{i,n}).$$

By the Cauchy–Schwarz inequality and property (iv),

$$|L_{n+m(n)}(f-p_n,x)| \leqslant \sqrt{n}\,K(c)\,E_n(f)\left\{\sum_{i=1}^{m(n)} \frac{\Phi_n^2(x)\,l_i^2(x)}{\Phi_n^2(y_{i,n})}\right\}^{1/2}.$$

From (10), the last inequality and the Lemma 1, we have

$$\int_{-1}^{1} w(x)\{L_{n+m(n)}(f,x)-p_n(x)\}^2\,dx$$
$$\leqslant nK^2(c)\,E_n^2(f)\,\frac{M(n)}{\rho}\int_{-1}^{1} w(x)\,dx, \tag{11}$$

therefore the hypothesis (9) implies that the last term tends to zero, and so the proof is complete.

3. On the Zeros of Jacobi Polynomials

Let us consider the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha,\beta>-1$, and for simplicity let us confine ourselves to the ultraspheric case $\alpha=\beta$. The same argument, with some more complication, may be used for $\alpha\neq\beta$.

Since we deal with the roots of $P_n^{(\alpha,\alpha)}(x)$, we shall indicate these polynomials by $P_n^{(\alpha)}(x)$ instead of the classical notation

$$P_n^{(\lambda)}(x)=\frac{\Gamma(\alpha+1)\,\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\,\Gamma(n+\alpha+1)}\,P_n^{(\alpha,\alpha)}(x)$$

$\alpha=\lambda-\frac{1}{2}$, $\lambda>-\frac{1}{2}$.

It is well known that the zeros of the polynomials $P_n^{(\alpha)}(x)$ with $\alpha<0$, as well as the zeros of $(1-x^2)(d/dx)\,P_n^{(\alpha)}(x)$ with $0\leqslant\alpha<1$, form a strongly normal matrix. Let the nodes matrix (5) be formed by the zeros of the polynomials

$$Q_{n'}(x)=(1-x^2)\,P_n^{(\alpha)}(x)\cdot\frac{d}{dx}\,P_n^{(\alpha)}(x)$$

or

$$Q_{n'}(x)=P_n^{(\alpha)}(x)\cdot\frac{d}{dx}\,P_n^{(\alpha)}(x),$$

where the nodes $x_{i,n}$'s and $y_{i,n}$'s are respectively the zeros of the factor polynomials $\Phi_n(x)$ and $\Psi_m(x)$ with:

case (a) $\Phi_n(x) = P_n^{(\alpha)}(x)$, $\Psi_m(x) = (1-x^2)\frac{d}{dx}P_n^{(\alpha)}(x)$ $0 \leqslant \alpha < 1$

case (b) $\Phi_n(x) = \frac{d}{dx}P_n^{(\alpha)}(x)$, $\Psi_m(x) = P_n^{(\alpha)}(x)$ $-1 < \alpha < 0$

case (c) $\Phi_n(x) = (1-x^2)\frac{d}{dx}P_n^{(\alpha)}(x)$, $\Psi_m(x) = P_n^{(\alpha)}(x)$ $-1 < \alpha < 0$.

In all cases, the matrix formed by the $y_{i,n}$'s nodes is strongly normal and, since $(d/dx)\,P_n^{(\alpha)}(x) = c(n)\,P_{n-1}^{(\alpha+1)}(x)$, the $x_{i,n}$'s nodes satisfy the condition (E). In order to apply the Theorem A, we have to investigate the behaviour of the function $M(n)$ in the three cases (a), (b), (c).

Case (a). In this case we have: deg $\Phi_n(x) = n$, deg $\Psi_m(x) = n+1$ and

$$M(n) = \frac{\int_{-1}^{1}(1-x^2)^\alpha\{P_n^{(\alpha)}(x)\}^2\,dx}{\min_i\{P_n^{(\alpha)}(y_{i,n})\}^2}.$$

Since the $y_{i,n}$'s are the zeros of $(1-x^2)(d/dx)\,P_n^{(\alpha)}(x)$, it is well known (see Szegö [5, p. 168]) that the sequence formed by the values $|P_n^{(\alpha)}(y_{i,n})|$ is decreasing in $[-1,0]$ and increasing in $[0,1]$. Therefore

$$\min_i\{P_n^{(\alpha)}(y_{i,n})\}^2 = \{P_n^{(\alpha)}(y_{[n/2]+1,n})\}^2 \cong n^{-1},$$

where the symbol $\cong$ refers to the limiting procedure $n\to\infty$ (see Szegö [5, p. 169]). Since also

$$\int_{-1}^{1}(1-x^2)^\alpha\{P_n^{(\alpha)}(x)\}^2\,dx \cong n^{-1}$$

we can conclude

$$M(n) = O(1)$$

and, in view of (9), we can state the following corollary.

COROLLARY 1. *For every $f(x)\in \mathrm{Lip}\,\gamma$, $\gamma > \frac{1}{2}$, the Lagrange interpolation polynomials $L_{2n+1}(f,x)$ based on the roots of $(1-x^2)\,P_n^{(\alpha)}(x)(d/dx)\,P_n^{(\alpha)}(x)$, $0\leqslant\alpha<1$, satisfy*

$$\lim_n \int_{-1}^{1}(1-x^2)^\alpha\{L_{2n+1}(f,x)-f(x)\}^2\,dx = 0.$$

Case (b). In this case we have $\deg \Phi_n = n - 1$, $\deg \Psi_m = n$ and

$$M(n) = \frac{\int_{-1}^{1} (1 - x^2)^{\alpha+1} \{(d/dx) P_n^{(\alpha)}(x)\}^2 \, dx}{\min_i \{(d/dx) P_n^{(\alpha)}(y_{i,n})\}^2}.$$

In order to evaluate the lower part of $M(n)$, let us prove the following lemma:

LEMMA 2. *For every* $-1 < \alpha < 0$, *the values of the function* $|(d/dx) P_n^{(\alpha)}(x)|$, *restricted to the roots of* $P_n^{(\alpha)}(x)$, *form a sequence which is decreasing in* $[-1, 0]$ *and increasing in* $[0, 1]$.

Proof. Let us consider the function

$$f(x) = n(n + 2\alpha + 1)(1 - x^2)^{\alpha} \{P_n^{(\alpha)}(x)\}^2 + (1 - x^2)^{\alpha+1} \left\{\frac{d}{dx} P_n^{(\alpha)}(x)\right\}^2 \tag{12}$$

and observe that

$$f(x) = (1 - x^2)^{\alpha+1} \left\{\frac{d}{dx} P_n^{(\alpha)}(x)\right\}^2$$

at the zeros of $P_n^{(\alpha)}(x)$. Since $P_n^{(\alpha)}(x)$ satisfies the differential equation

$$(1 - x^2) y'' - 2(\alpha + 1) xy' + n(n + 2\alpha + 1) y = 0, \tag{13}$$

by differentiating (12) we obtain

$$f'(x) = x \left[2 \left\{\frac{d}{dx} P_n^{(\alpha)}(x)\right\}^2 (1 - x^2)^{\alpha}(\alpha + 1) - 2\alpha n(n + 2\alpha + 1)(1 - x^2)^{\alpha-1} \{P_n^{(\alpha)}(x)\}^2\right]$$

and hence

$$\operatorname{sgn} x = \operatorname{sgn} f'(x).$$

Therefore, on the zeros of $P_n^{(\alpha)}(x)$, the function $(1 - x^2)^{\alpha+1} \{(d/dx) P_n^{(\alpha)}(x)\}^2$ is decreasing in $[-1, 0]$ and increasing in $[0, 1]$; hence, a fortiori, the function $(d/dx) P_n^{(\alpha)}(x)$ has the same behaviour.

Analogously to the preceding case,

$$M(n) = \frac{\int_{-1}^{1} (1-x^2)^{\alpha+1} \{(d/dx)\, P_n^{(\alpha)}(x)\}^2\, dx}{\{(d/dx)\, P_n^{(\alpha)}(y_{[(n+1)/2],n})\}^2}$$

$$= \frac{\int_{-1}^{1} (1-x^2)^{\alpha+1} \{P_{n-1}^{\alpha+1}(x)\}^2\, dx}{\{P_{n-1}^{\alpha+1}(y_{[(n+1)/2],n})\}^2} = O(1)$$

and hence the following corollary holds:

COROLLARY 2. *For every* $f(x) \in \operatorname{Lip} \gamma$, $\gamma > \frac{1}{2}$ *the Lagrange interpolation polynomials* $L_{2n-1}(f, x)$ *based on the roots of* $P_n^{(\alpha)}(x)(d/dx)\, P_n^{(\alpha)}(x)$, $-1 < \alpha < 0$, *satisfy*

$$\lim_n \int_{-1}^{1} (1-x^2)^{\alpha+1} \{L_{2n-1}(f, x) - f(x)\}^2\, dx = 0.$$

Case (c). In this case we have $\deg \Phi_n = n+1$, $\deg \Psi_m = n$ and

$$M(n) = \frac{\int_{-1}^{1} (1-x^2)^{\alpha} \{(1-x^2)(d/dx)\, P_n^{(\alpha)}(x)\}^2\, dx}{\min_i \{(1-y_{i,n}^2)(d/dx)\, P_n^{(\alpha)}(y_{i,n})\}^2}.$$

Let us prove the following analogue to Lemma 2:

LEMMA 3. *For every* $-1 < \alpha < 0$, *the values of the function* $|(1-x^2)(d/dx)\, P_n^{(\alpha)}(x)|$, *restricted to the roots of* $P_n^{(\alpha)}(x)$, *form a sequence which is increasing in* $[-1, 0]$ *and decreasing in* $[0, 1]$.

Proof. Let us consider the function $g(x) = (1-x^2)\, f(x)$, $f(x)$ defined in (12), and observe that it takes the values of $(1-x^2)^{\alpha+2}\{(d/dx)\, P_n^{(\alpha)}(x)\}^2$ on the zeros of $P_n^{(\alpha)}(x)$. The derivative of $g(x)$, after some manipulation, taking into account that $P_n^{(\alpha)}(x)$ satisfies the differential equation (13), becomes

$$g'(x) = x \left[2\alpha(1-x^2)^{\alpha+1} \left\{ \frac{d}{dx} P_n^{(\alpha)}(x) \right\}^2 - 2(\alpha+1)\, n(n+2\alpha+1)(1-x^2)\{P_n^{(\alpha)}(x)\}^2 \right]$$

and then

$$\operatorname{sgn} x = -\operatorname{sgn} g'(x).$$

Therefore, on the roots of $P_n^{(\alpha)}(x)$, the function

$$g(x) = (1-x^2)^{\alpha} \left\{ (1-x^2) \frac{d}{dx} P_n^{(\alpha)}(x) \right\}^2$$

and, a fortiori, the function

$$\left\{(1-x^2)\frac{d}{dx}P_n^{(\alpha)}(x)\right\}^2,$$

is increasing in $[-1,0]$ and decreasing in $[0,1]$. So the lemma is proved. Then

$$\min_i\left\{(1-y_{i,n}^2)\frac{d}{dx}P_n^{(\alpha)}(y_{i,n})\right\}^2=\left\{(1-y_{1,n}^2)\frac{d}{dx}P_n^{(\alpha)}(y_{1,n})\right\}^2$$

and

$$M(n)=\frac{\int_{-1}^{1}(1-x^2)^{\alpha}\{(1-x^2)(d/dx)P_n^{(\alpha)}(x)\}^2\,dx}{\{(1-y_{1,n}^2)(d/dx)P_n^{(\alpha)}(y_{1,n})\}^2}$$
$$\leqslant\frac{\int_{-1}^{1}(1-x^2)^{\alpha+1}\{(d/dx)P_n^{(\alpha)}(x)\}^2\,dx}{\{(1-y_{1,n}^2)(d/dx)P_n^{(\alpha)}(y_{1,n})\}^2},$$

which, by the recurrence formulae (4.5.5) and (4.5.7) in Szegö [5], is equal to

$$=\frac{((n+2\alpha+1)^2/4)\int_{-1}^{1}(1-x^2)^{\alpha+1}\{P_{n-1}^{\alpha+1}(x)\}^2\,dx}{(n+\alpha)^2\{P_{n-1}^{(\alpha)}(y_{1,n})\}^2}.$$

Since $\vartheta_{1,n}\cong n^{-1}$ (see Theorem (8.2.1) in Szegö [5]), where $y_{1,n}=\cos\vartheta_{1,n}$, the "Hilb's type" formula (8.21.17) in Szegö [5], leads to $P_n^{(\alpha)}(y_{1,n})\cong n^2$. Therefore

$$M(n)\cong\frac{n^{-1}}{n^{2\alpha}}\cong n^{-1-2\alpha}$$

and the following corollary holds:

COROLLARY 3. *For every function $f(x)\in\mathrm{Lip}\,\gamma$, the Lagrange interpolation polynomials $L_{2n+1}(f,x)$ based on the roots of $(1-x^2)P_n^{(\alpha)}(x)(d/dx)P_n^{(\alpha)}(x)$, $-1<\alpha<0$, satisfy*

$$\lim_{n\to\infty}\int_{-1}^{1}(1-x^2)^{\alpha}\{L_{2n+1}(f,x)-f(x)\}^2\,dx=0$$

provided $\gamma>-\alpha$.

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